Olimpiada de Matematică –etapa locală- Galaţi

24 februarie 2019

Clasa a XI-a

Barem de evaluare

- **Pentru orice soluţie corectă, chiar dacă este diferită de cea din barem, se acordă punctajul maxim corespunzător.**
- **Nu se acordă fracţiuni de punct, dar se pot acorda punctaje intermediare pentru rezolvări parţiale, în limitele punctajului indicat în barem.**

Nr. problemei	**Soluţie, rezolvare**	**Punctaj**
1.	$a)\ A-B=\begin{pmatrix}1&0&0\\0&0&0\\0&0&2\end{pmatrix}$; rang $(A-B)=2.$	**2p**
	$b)\ A\cdot T-T\cdot A=B\cdot T-T\cdot B\Leftrightarrow A\cdot T-BT=T\cdot A-T\cdot B\Leftrightarrow$ $(A-B)\cdot T=T\cdot(A-B);$	**2p**
	$(A-B)\cdot T=\begin{pmatrix}1&0&0\\0&0&0\\0&0&2\end{pmatrix}\cdot\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}=\begin{pmatrix}a&b&c\\0&0&0\\2g&2h&2i\end{pmatrix};$	**1p**
	$T\cdot(A-B)=\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}\cdot\begin{pmatrix}1&0&0\\0&0&0\\0&0&2\end{pmatrix}=\begin{pmatrix}a&0&2c\\d&0&2f\\g&0&2i\end{pmatrix}\Rightarrow T=\begin{pmatrix}a&0&0\\0&e&0\\0&0&i\end{pmatrix};$	**1p**
	Notăm $\mathrm{T}=\begin{pmatrix}a&0&0\\0&b&0\\0&0&c\end{pmatrix},\ a,\ b,\ c\in\mathbb{R}$ $\mathrm{T}^3=\begin{pmatrix}a^3&0&0\\0&b^3&0\\0&0&c^3\end{pmatrix}=I_3\Rightarrow\begin{cases}a^3=1\\b^3=1\\c^3=1\\a,\ b,\ c\in\mathbb{R}\end{cases}\Rightarrow a=b=c=1\Rightarrow T=I_3.$	**1p**

	$a)$ $\mathrm{A}\in \mathrm{E}\Rightarrow a+d=b+c;$ det A=$a\cdot d$ - $b\cdot c$; $\mathrm{A}^2=\begin{pmatrix} a^2+b\cdot c & b\cdot(a+d) \\ c\cdot(a+d) & d^2+b\cdot c \end{pmatrix}\in \mathrm{E}\Rightarrow$	1p
	$a^2+b\cdot c+d^2+b\cdot c=b\cdot(a+d)+c\cdot(a+d)\Leftrightarrow$ $b\cdot c=a\cdot d\Rightarrow \det \mathrm{A}=0.$	1p
	$b)$ Demonstrăm că $(\forall)\mathrm{A},\ \mathrm{B}\in \mathrm{E}$ şi $\alpha,\beta\in \mathbb{R}$ $\Rightarrow \alpha\cdot \mathrm{A}+\beta\cdot \mathrm{B}\in \mathrm{E}$ Fie A=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in E\Rightarrow a+d=b+c \left\vert \overset{\cdot\alpha}{\Rightarrow} \right. \alpha\cdot(a+d)=\alpha\cdot(b+c)\quad (1)$ $\mathrm{B}=\begin{pmatrix} x & y \\ z & t \end{pmatrix}\in \mathrm{E}\Rightarrow x+t=y+z \left\vert \overset{\cdot\beta}{\Rightarrow} \right. \beta\cdot(x+t)=\beta\cdot(y+z)\qquad (2)$	1p
2.	Adunăm (1) şi $(2)\Rightarrow \alpha\cdot(a+d)+\beta\cdot(x+t)=\alpha\cdot(b+c)+\beta\cdot(y+z)\quad (3)$ Dar $\alpha\cdot \mathrm{A}+\beta\cdot \mathrm{B}=\begin{pmatrix} \alpha\cdot a+\beta\cdot x & \alpha\cdot b+\beta\cdot y \\ \alpha\cdot c+\beta\cdot z & \alpha\cdot d+\beta\cdot t \end{pmatrix}\qquad (4)$ Din (3) şi $(4)\Rightarrow \alpha\cdot \mathrm{A}+\beta\cdot \mathrm{B}\in \mathrm{E}$	1p
	c) Presupunem că $\det\mathrm{A}\neq 0$; $\mathrm{A}^2-\mathrm{TrA}\cdot \mathrm{A}+\det \mathrm{A}\cdot I_2=\mathrm{O}_2 \left\vert \overset{\cdot \mathrm{A}^{n-1}}{\Rightarrow} \right. \mathrm{A}^{n+1}-\mathrm{TrA}\cdot \mathrm{A}^n+\det \mathrm{A}\cdot \mathrm{A}^{n-1}=\mathrm{O}_2;$ $\left.\begin{array}{l} \mathrm{A}^{n+1}-\mathrm{TrA}\cdot \mathrm{A}^n+\det \mathrm{A}\cdot \mathrm{A}^{n-1}=\mathrm{O}_2 \\ \text{Dar } \mathrm{A}^{n+1} \text{şi } \mathrm{A}^n\in \mathrm{E} \end{array}\right\}\Rightarrow \mathrm{A}^{n-1}\in \mathrm{E};$	**1p**
	Analog $\mathrm{A}^n-\mathrm{TrA}\cdot \mathrm{A}^{n-1}+\det \mathrm{A}\cdot \mathrm{A}^{n-2}=\mathrm{O}_2;$ $\left.\begin{array}{l} \text{Cum } \mathrm{A}^n \text{ şi } \mathrm{A}^{n-1}\in E \\ \det \mathrm{A}\neq 0 \end{array}\right\}\Rightarrow \mathrm{A}^{n-2}\in E$ ş.a.m.d $\Rightarrow \mathrm{A}^2$ şi $\mathrm{A}\in \mathrm{E}\Rightarrow$ det A=0.	**2p**

3.

Soluţie:

Fie $y=x-2 \to 0 \Rightarrow x=y+2$; **1p**

$$\lim_{y\to 0}\frac{\sqrt{4y+9}-\sqrt[3]{y+8}-\sqrt[4]{y+1}}{9\cdot 3^y+16\cdot 4^y-25\cdot 5^y}=$$

1p

$$\lim_{y\to 0}\frac{3\cdot\sqrt{\frac{4}{9}\cdot y+1}-2\cdot\sqrt[3]{\frac{y}{8}+1}-\sqrt[4]{y+1}}{y}\cdot\frac{1}{\frac{9\cdot 3^y}{y}+\frac{16\cdot 4^y}{y}-\frac{25\cdot 5^y}{y}}=$$

1p

$$\lim_{y\to 0}\left[\frac{3\left(\sqrt{\frac{4}{9}\cdot y+1}-1\right)}{y}-2\cdot\frac{\sqrt[3]{\frac{y}{8}+1}-1}{y}-\frac{\sqrt[4]{y+1}-1}{y}\right]\cdot$$

$$\lim_{y\to 0}\frac{1}{9\cdot\frac{3^y-1}{y}+16\cdot\frac{4^y-1}{y}-25\cdot\frac{5^y-1}{y}}=$$

2p

$$\left(3\cdot\lim_{y\to 0}\frac{\left(\sqrt{\frac{4}{9}\cdot y+1}-1\right)}{\frac{4}{9}\cdot y}\cdot\frac{4}{9}-2\cdot\lim_{y\to 0}\frac{\sqrt[3]{\frac{y}{8}+1}-1}{\frac{1}{8}y}\cdot\frac{1}{8}-\lim_{y\to 0}\frac{\sqrt[4]{y+1}-1}{y}\right)\cdot$$

$$\cdot\frac{1}{9\cdot\ln 3+16\cdot\ln 4-25\cdot\ln 5}=$$

$$\left(3\cdot\frac{1}{2}\cdot\frac{4}{9}-2\cdot\frac{1}{3}\cdot\frac{1}{8}-\frac{1}{4}\right)\cdot\frac{1}{9\cdot\ln 3+16\cdot\ln 4-25\cdot\ln 5}=$$

2p

$$\frac{1}{3\cdot\left(9\cdot\ln 3+32\cdot\ln 2-25\cdot\ln 5\right)}.$$

4.	$a)$ Fie şirurile de numere reale: $a_n = 1+2^a+3^a+...+n^a,\ b_n = n^{a+1},\ a>0;$ Se aplică teorema Stolz-Cezaro$'$: $\lim\limits_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim\limits_{n\to\infty}\frac{(n+1)^a}{(n+1)^{a+1}-n^{a+1}} = \lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^a\cdot\frac{1}{n}\cdot\frac{1}{\left(\frac{n+1}{n}\right)^{a+1}-1} =$ $\lim\limits_{n\to\infty}\left(\frac{n+1}{n}\right)^a\cdot\frac{1}{\frac{\left(1+\frac{1}{n}\right)^{a+1}-1}{\frac{1}{n}}} = \frac{1}{a+1};$ S-a folosit limita remarcabilă: $\lim\limits_{x\to 0}\frac{(1+x)^r-1}{x} = r,\ r>0.$ $b)$ Conform punctului a) avem $\lim\limits_{n\to\infty}\frac{1+\sqrt[4]{2}+\sqrt[4]{3}+...+\sqrt[4]{n}}{n^{1+\frac{1}{4}}} = \frac{4}{5};$ Rămâne să calculăm $\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}}\cdot\frac{1}{\left(\sqrt[4]{(n+1)^2+1}-\sqrt[4]{n^2+1}\right)} =$ $\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}}\cdot\frac{\left(\sqrt[4]{(n+1)^2+1}\right)^3+\left(\sqrt[4]{(n+1)^2+1}\right)^2\cdot\sqrt[4]{n^2+1}+\left(\sqrt[4]{(n+1)^2+1}\right)\cdot\left(\sqrt[4]{n^2+1}\right)^2+\left(\sqrt[4]{n^2+1}\right)^3}{(n+1)^2-n^2} =$ $\frac{4}{2}=2;$ Atunci $\lim\limits_{n\to\infty}\frac{1+\sqrt[4]{2}+\sqrt[4]{3}+...+\sqrt[4]{n}}{n\cdot\sqrt[4]{n^3}\cdot\left(\sqrt[4]{(n+1)^2+1}-\sqrt[4]{n^2+1}\right)} = \frac{4}{5}\cdot 2 = \frac{8}{5}.$	**1p** **1p** **1p** **2p** **1p** **1p**